POSTERIOR ANALYSIS OF NAKAGAMI DISTRIBUTION UNDER DIFFERENT LOSS FUNCTIONS

Wasif Yaseen and *Navid Feroze[1]
College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan
[1]Department of Statistics, Government Post Graduate College Muzaffarabad, Azad Kashmir, Pakistan
*Email: navidferoz@gmail.com

ABSTRACT: *Nakagami distribution is a flexible life time distribution that can be used in the analysis of lifetime data and in problems related to the modeling of failure processes. It has applications in attenuation of wireless signals traversing multiple paths, deriving unit hydrographs in hydrology, medical imaging studies etc. We have considered different priors for the analysis of the scale parameter of Nakagami distribution under various loss functions. It has been assessed that the performance of inverse gamma prior is better than uniform prior and weighted balanced loss function (WBLF) performs the best among all loss functions.*

1. INTRODUCTION

Nakagami distribution can be considered as a flexible life time distribution. It was primarily proposed for modeling the fading of radio signals. Although, the model may also offer a good fit to some failure time data. It has been used to model attenuation of wireless signals traversing multiple paths. The Nakgami distribution is comprehensively used to model the fading of radio signals and other areas of communicational engineering. It may also be used in hydrology in order to derive the unit hydrographs. The applications of the distribution can also be found in medical imaging studies to model the ultrasounds especially in Echo (heart efficiency test) and in modeling high-frequency seismogram envelopes. The distribution may also be employed to model failure times of a variety of products (and electrical components) such as ball bearing, vacuum tubes, electrical insulation. It is also widely considered in biomedical fields, such as to model the time to the occurrence of tumors and appearance of lung cancer. This distribution is extensively used in reliability theory, reliability engineering and to model the constant hazard rate portion because of its memory less property. Moreover, it is very convenient because it is so simple to add failure rates in a reliability model.

In physics, if we observe a gas at a fixed temperature and pressure in a uniform gravitational field, then the height of the various molecules can be modeled by a Nakagami distribution. Interestingly, the Nakagami distribution is the best distribution to check the reliability of electrical components as compare to the Gamma, Weibull and lognormal distribution.

1.1 Probability Density Function of Nakagami Distribution

The probability density function of the distribution is given as:

$$f\left(x;\theta\right)=\frac{2\lambda^{\lambda}x^{2\lambda-1}}{\Gamma\left(\lambda\right)\theta^{\lambda}}\exp\left[\frac{-\lambda x^{2}}{\theta}\right], \qquad \theta>0$$

Where, $\lambda \geq 0.5$ is the shape parameter and $\theta > 0$ is scale parameter. It collapses to Rayleigh distribution when λ =1 and half normal distribution λ =0.5.

Much work has been made on Nakagami distribution. The use of the distribution can be seen in a number of scientific fields including Telecommunication Engineering and medical research. Some of the researchers have used this distribution to model attenuation of wireless signals traversing multiple paths, bit error rate (BER) performance of an M-branch maximal-ratio combiner (MRC), spatial-chromatic distribution of images, multipath faded envelope in wireless channels, vivo breast data, multimedia and ultrasound data in medical imaging studies. The real life applications of the distribution can be found from the contributions of: [1, 2,3, 4, 5]. The Nakagami distribution has remained under the consideration of the classical statisticians. The significant piece of work, on the distribution, has been done under frequentist approach such as maximum-likelihood estimation, direct-sum decomposition principle, correlated Nakagami process, probability density function of the sum and the difference of two correlated squared Nakagami variates, backscatter analysis based on generalized entropies and neural function approximation, compressed logarithmic computation and bootstrap bias-corrected maximum likelihood estimation. Some important contributions in this regard are as follow: [6,7,8, 9, 10, 11,12].

However, the Nakagami distribution has not been considered frequently for the analysis under the Bayesian framework. Therefore, we have considered the Bayesian analysis of the distribution under different priors and loss functions in order to find the most appropriate combination of loss function and prior for the estimation of the scale parameter of the distribution.

The authors considering the Bayesian analysis of the probability distributions include: [13, 14, 15, 16, 17].

2. MATERIAL AND METHODS

This section covers the material and methods for the study.

2.1 Informative and Uninformative Priors

Following informative and non-informative priors have been used for analysis of the scale parameter of the Nakagami distribution.

2.1.1 Inverse Gamma Prior

The inverse gamma can be presented as:

$$P\left(\theta\right)=\frac{b^{-c}\theta^{-c-1}}{\Gamma\left(c\right)}\exp\left[\frac{-b}{\theta}\right], \qquad \theta>0$$

2.1.2 Uniform Prior

One of the most famous non-informative priors is a uniform prior, it can be given as:

$$P\left(\theta\right)=K$$

2.2 Derivation Of Posterior Estimates

This section contains the derivation of the Bayes estimators and posterior risks under different priors and loss functions.

2.2.1 Joint Distribution of the Sample and Scale Parameter Θ

The likelihood function of Nakagami distribution

$$L(\theta|\mathrm{x}) \propto \theta^{-n\lambda} \exp\left(\frac{-\lambda}{\theta \sum_{i=1}^{n} x_i^2} \right)$$

2.2.2 Posterior Distribution using Uniform Prior

The posterior distribution under the assumption of Uniform prior is:

$$P(\theta|\mathrm{x}) = \frac{\left(\lambda \sum_{i=1}^{n} x_i^2\right)^{n\lambda-1}}{\Gamma(n\lambda-1)} \theta^{-n\lambda} \exp\left(\frac{-\lambda}{\theta \sum_{i=1}^{n} x_i^2} \right), \theta > 0$$

2.2.3 Bayesian Estimation under Three Loss Functions

In Bayesian analysis the comparisons among different estimators are made on the basis of loss functions. We have used following loss functions for the derivations of Bayes estimates and corresponding posterior risks. Further, we find below some important results that are needed for the derivation of Bayes and posterior risks under various loss functions.

i) $$\mathrm{E}(\theta) = \frac{\lambda \sum_{i=1}^{n} x_i^2}{n\lambda - 2}$$

ii) $$\mathrm{E}(\theta^2) = \frac{\left(\lambda \sum_{i=1}^{n} x_i^2\right)^2}{(n\lambda-2)(n\lambda-3)}$$

iii) $$\mathrm{E}(\theta^{-1}) = \frac{n\lambda - 1}{\lambda \sum_{i=1}^{n} x_i^2}$$

2.2.3.1 Weighted Loss Function (WLF)

The formulas for Bayes estimate and corresponding posterior risk under WLF are as under:

The Bayes estimator under WLF is:

$$\theta_{WLF} = \left\{\mathrm{E}(\theta^{-1})\right\}^{-1} = \frac{\lambda \sum_{i=1}^{n} x_i^2}{n\lambda - 1}$$

The posterior risk of the Bayes estimator under WLF is:

$$\rho(\theta_{WLF}) = \mathrm{E}(\theta) - \theta_{WLF} = \frac{\lambda \sum_{i=1}^{n} x_i^2}{(n\lambda-1)(n\lambda-2)}$$

2.2.3.2 Weighted Balanced Loss Function (WBLF)

The Weighted Balanced loss function is defined as: $L(\theta_{WBLF}, \theta) = \left(\frac{\theta_{WBLF} - \theta}{\theta_{WBLF}} \right)^2$

The formulas for Bayes estimate and corresponding posterior risk under WBLF are as under:

The Bayes estimator under WBLF is:

$$\theta_{WBLF} = \mathrm{E}(\theta^2)\left\{\mathrm{E}(\theta)\right\}^{-1} = \frac{\lambda \sum_{i=1}^{n} x_i^2}{n\lambda - 3}$$

The Bayes risk under WBLF is:

$$\rho(\theta_{WBLF}) = 1 - \frac{\left\{\mathrm{E}(\theta)\right\}^2}{\mathrm{E}(\theta^2)} = 1 - \frac{1}{n\lambda - 2}$$

2.2.3.3 Precautionary Loss Function (PLF)

The precautionary loss function (PLF) can be presented as:

$$L(\theta_{PLF}, \theta) = \frac{(\theta_{PLF} - \theta)^2}{\theta_{PLF}}$$

The formulas for Bayes estimate and corresponding posterior risk under PLF are as under:

The Bayes estimator under PLF is:

$$\theta_{PLF} = \left\{\mathrm{E}(\theta^2)\right\}^{\frac{1}{2}} = \frac{\lambda \sum_{i=1}^{n} x_i^2}{\sqrt{(n\lambda-2)(n\lambda-3)}}$$

The posterior risk of the Bayes estimator under PLF is:

$$\rho(\theta_{PLF}) = 2\left\{\theta_{PLF} - \mathrm{E}(\theta)\right\}$$

$$= 2\left\{ \frac{\lambda \sum_{i=1}^{n} x_i^2}{\sqrt{(n\lambda-2)(n\lambda-3)}} - \frac{\lambda \sum_{i=1}^{n} x_i^2}{n\lambda - 2} \right\}$$

The Bayesian estimates under inverse gamma prior can be obtained with little modifications.

3. RESULTS AND DISCUSSIONS

A simulation study has been conducted to evaluate the behavior and performance of different estimators. A comparison in terms of magnitude of posterior risks is needed to check whether an estimator is inadmissible under some loss function or prior distribution. The samples have been simulated for n = 5, 20, 40, 100, 150, 250 and 400 using $(\lambda, \theta) \in \left\{ \begin{matrix} (1, 0.5), (1,1), (1,1.5), (1,2), \\ (2, 0.5), (2,1), (2,1.5), (2,2) \end{matrix} \right\}$ under

Table 3.1.1: Bayesian estimates under uniform prior using WLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.8280	1.6922	2.4869	3.2906	0.6287	1.2511	1.8676	2.5011
20	0.5534	1.1337	1.6556	2.1870	0.5268	1.0450	1.6103	2.1047
40	0.5235	1.0710	1.5738	2.0744	0.5130	1.0226	1.5659	2.0451
100	0.5078	1.0392	1.5253	2.0125	0.5058	1.0057	1.5391	2.0154
150	0.5035	1.0304	1.5149	1.9993	0.5039	1.0019	1.5344	2.0101
250	0.5022	1.0272	1.5072	1.9853	0.5027	1.0000	1.5301	2.0060
400	0.5003	1.0238	1.5001	1.9811	0.5021	0.9982	1.5285	2.0026

Table 3.1.2: Bayesian estimates under uniform prior using WBLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.4968	1.0153	1.4921	1.9744	0.5029	1.0008	1.4941	2.0008
20	0.4980	1.0203	1.4900	1.9683	0.5005	0.9927	1.5297	1.9995
40	0.4973	1.0175	1.4951	1.9707	0.5002	0.9971	1.5268	1.9940
100	0.4976	1.0184	1.4948	1.9723	0.5007	0.9956	1.5237	1.9952
150	0.4968	1.0167	1.4947	1.9726	0.5005	0.9952	1.5241	1.9967
250	0.4982	1.0190	1.4951	1.9694	0.5007	0.9960	1.5240	1.9980
400	0.4978	1.0187	1.4926	1.9712	0.5009	0.9957	1.5246	1.9976

Table 3.1.3: Bayesian estimates under uniform prior using PLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	1.0141	2.0725	3.0458	4.0301	0.6721	1.3374	1.9966	2.6737
20	0.5694	1.1665	1.7036	2.2504	0.5339	1.0590	1.6319	2.1329
40	0.5305	1.0854	1.5949	2.1022	0.5163	1.0293	1.5760	2.0584
100	0.5104	1.0445	1.5332	2.0229	0.5071	1.0083	1.5430	2.0205
150	0.5052	1.0339	1.5201	2.0061	0.5047	1.0035	1.5369	2.0135
250	0.5032	1.0293	1.5102	1.9893	0.5032	1.0010	1.5317	2.0081
400	0.5009	1.0251	1.5020	1.9836	0.5025	0.9988	1.5294	2.0039

Table 3.1.2 : Posterior Risks under uniform prior using WLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.0966	0.2194	0.2890	0.2974	0.0599	0.1068	0.5289	0.1490
20	0.0182	0.0765	0.1634	0.2850	0.0074	0.0292	0.0692	0.1185
40	0.0073	0.0307	0.0662	0.1151	0.0033	0.0133	0.0311	0.0531
100	0.0026	0.0109	0.0234	0.0407	0.0013	0.0050	0.0117	0.0200
150	0.0017	0.0070	0.0152	0.0264	0.0008	0.0033	0.0077	0.0132
250	0.0010	0.0041	0.0089	0.0155	0.0005	0.0019	0.0046	0.0078
400	0.0006	0.0026	0.0055	0.0096	0.0003	0.0012	0.0028	0.0049

Table 3.1.5 : Posterior Risks under uniform prior using WBLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.0567	0.0853	0.0784	0.1077	0.0132	0.0263	0.0393	0.0526
20	0.0105	0.0215	0.0313	0.0414	0.0046	0.0092	0.0141	0.0184
40	0.0046	0.0094	0.0138	0.0182	0.0022	0.0043	0.0066	0.0086
100	0.0017	0.0035	0.0051	0.0068	0.0008	0.0017	0.0025	0.0033
150	0.0011	0.0023	0.0034	0.0044	0.0006	0.0010	0.0017	0.0022
250	0.0007	0.0014	0.0020	0.0026	0.0003	0.0006	0.0010	0.0013
400	0.0004	0.0008	0.0012	0.0016	0.0002	0.0003	0.0006	0.0007

Table 3.1.6 : Posterior Risks under uniform prior using PLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.1825	0.3730	0.5482	0.7254	0.0426	0.0847	0.1265	0.1694
20	0.0157	0.0322	0.0471	0.0622	0.0069	0.0138	0.0212	0.0277
40	0.0069	0.0141	0.0207	0.0273	0.0033	0.0065	0.0099	0.0130
100	0.0026	0.0052	0.0077	0.0102	0.0013	0.0025	0.0038	0.0050
150	0.0017	0.0034	0.0050	0.0067	0.0008	0.0016	0.0025	0.0033
250	0.0011	0.0020	0.0030	0.0039	0.0005	0.0010	0.0015	0.0020
400	0.0006	0.0013	0.0019	0.0024	0.0003	0.0005	0.0009	0.0012

Table 3.2.3: Bayesian estimates under inverse gamma prior using WLF

n	λ= 1. θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.6987	1.2263	1.6973	2.1737	0.6012	1.0999	1.5934	2.0996
20	0.5496	1.0735	1.5532	2.0260	0.5278	1.0271	1.5169	2.0326
40	0.5250	1.0471	1.5231	2.0123	0.5155	1.0133	1.5113	2.0194
100	0.5100	1.0332	1.5106	1.9867	0.5081	1.0044	1.4994	2.0092
150	0.5064	1.0285	1.5061	1.9870	0.5061	1.0044	1.4980	2.0093
250	0.5036	1.0258	1.5039	1.9802	0.5047	1.0016	1.4975	2.0069
400	0.5020	1.0257	1.5029	1.9812	0.5042	1.0006	1.4960	2.0072

Table 3.2.2: Bayesian estimates under inverse gamma prior using WBLF

n	λ= 1. θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.4991	0.8759	1.2124	1.5527	0.5010	0.9166	1.3278	1.7497
20	0.4997	0.9760	1.4120	1.8418	0.5026	0.9782	1.4447	1.9358
40	0.5000	0.9972	1.4506	1.9165	0.5029	0.9886	1.4744	1.9702
100	0.5000	1.0129	1.4810	1.9477	0.5030	0.9944	1.4845	1.9893
150	0.4997	1.0149	1.4863	1.9608	0.5027	0.9978	1.4881	1.9960
250	0.4996	1.0176	1.4920	1.9645	0.5026	0.9976	1.4915	1.9989
400	0.4995	1.0206	1.4954	1.9714	0.5029	0.9981	1.4923	2.0022

Table 3.2.3: Bayesian estimates under inverse gamma prior using PLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.7812	1.3710	1.8977	2.4303	0.6338	1.1594	1.6796	2.2132
20	0.5639	1.1014	1.5935	2.0786	0.5345	1.0402	1.5362	2.0585
40	0.5317	1.0604	1.5425	2.0380	0.5188	1.0197	1.5208	2.0322
100	0.5126	1.0384	1.5182	1.9967	0.5094	1.0069	1.5032	2.0143
150	0.5081	1.0319	1.5111	1.9936	0.5069	1.0061	1.5005	2.0127
250	0.5046	1.0278	1.5070	1.9842	0.5052	1.0026	1.4990	2.0089
400	0.5026	1.0270	1.5048	1.9837	0.5045	1.0013	1.4969	2.0085

Table 3.2.4: Posterior Risks under Inverse Gamma Prior Using WLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.0785	0.2005	0.1972	0.2180	0.0556	0.0991	0.2939	0.1126
20	0.0159	0.0606	0.1271	0.2165	0.0070	0.0265	0.0579	0.1040
40	0.0069	0.0275	0.0583	0.1017	0.0026	0.0126	0.0280	0.0501
100	0.0025	0.0104	0.0223	0.0385	0.0012	0.0049	0.0109	0.0195
150	0.0016	0.0068	0.0147	0.0255	0.0007	0.0032	0.0072	0.0130
250	0.0009	0.0040	0.0087	0.0151	0.0004	0.0018	0.0043	0.0077
400	0.0006	0.0025	0.0054	0.0094	0.0003	0.0011	0.0027	0.0048

Table 3.2.5: Posterior Risks under Inverse Gamma Prior Using WBLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.0539	0.0810	0.0745	0.1023	0.0125	0.0250	0.0373	0.0500
20	0.0100	0.0204	0.0297	0.0393	0.0044	0.0087	0.0134	0.0175
40	0.0044	0.0089	0.0131	0.0173	0.0021	0.0041	0.0063	0.0082
100	0.0016	0.0033	0.0048	0.0065	0.0008	0.0016	0.0024	0.0031
150	0.0010	0.0022	0.0032	0.0042	0.0006	0.0010	0.0016	0.0021
250	0.0006	0.0012	0.0017	0.0023	0.0003	0.0005	0.0009	0.0011
400	0.0003	0.0007	0.0010	0.0014	0.0002	0.0003	0.0005	0.0006

Table 3.2.6: Posterior Risks under Inverse Gamma Prior Using PLF

n	λ= 1, θ = 0.5	λ= 1, θ = 1	λ= 1, θ = 1.5	λ= 1, θ = 2	λ= 2, θ = 0.5	λ= 2, θ = 1	λ= 2, θ = 1.5	λ= 2, θ = 2
5	0.1633	0.2866	0.3967	0.5080	0.0344	0.0782	0.1006	0.1249
20	0.0140	0.0247	0.0341	0.0436	0.0056	0.0127	0.0169	0.0204
40	0.0062	0.0108	0.0150	0.0191	0.0027	0.0060	0.0079	0.0096
100	0.0023	0.0040	0.0056	0.0071	0.0010	0.0023	0.0030	0.0037
150	0.0015	0.0026	0.0036	0.0047	0.0006	0.0015	0.0020	0.0024
250	0.0010	0.0015	0.0022	0.0027	0.0004	0.0009	0.0012	0.0015
400	0.0005	0.0010	0.0014	0.0017	0.0002	0.0005	0.0007	0.0009

1000 replications. The resultant Bayes estimates and posterior risks under different priors and loss functions are presented in the tables below.

3.1 Simulation Results for Bayesian Estimates under Uniform Prior

The Bayesian estimates and their posterior risks under uniform prior are presented in the following tables.

3.2 Simulation Results for Bayesian Estimates under Inverse Gamma Prior

The Bayesian estimates and their posterior risks under inverse gamma prior are presented in the following tables.

4. CONCLUSION

Nakagami distribution has a wide range of applications in communicational engineering and medical studies. A number of contributions of Nakagami distribution appear in classical statistics. In this paper, the Bayesian estimators of the scale parameter of Nakagami distribution are obtained. We have considered the uniform and inverse gamma priors for the derivation of the posterior distribution of mentioned parameter. The three loss functions namely weighted loss function; weighted balanced loss function and precautionary loss function have been used for estimation. The performance of an estimator is assessed on the basis of its relative posterior risk. The Monte Carlo Simulations are used to compare the performance of the estimators. The salient results of this analysis are as follow.

4.1 Sample Size

The posterior risks based on both priors and for all loss functions, relating to the scale parameter of a Nakagami distribution, expectedly decrease with the increase in sample size.

4.2 Convergence of Bayes Estimates

The Bayes estimates tend to be close to the original values as sample size increases. Hence the Bayes estimates are consistent which is in accordance with the theory.

4.3 Posterior Risks

The magnitude of the posterior risks is directly proportional to the true parametric values, while it is inversely proportional to the sample size. This property is common under all priors and all loss functions. The amounts of posterior risks are smaller for greater values of the shape parameters of the Nakagami distribution (keeping the values of the scale parameter specified).

4.4 Priors

Using the uniform prior, the posterior risk increases with increase in the value of θ whatever the value of λ may be. At the same level of θ, the posterior risk decreases for Nakagami distribution with a larger λ. On using the Inverse Gamma Prior with the hyper parameters taking the values b = 1 & c = 2, b = 0.5 & c = 3 and b = 3 & c = 0.5, and fixed λ, it is found that the posterior risk increases when the Bayesian estimator of a larger scale parameter is needed. For the same unknown θ value, the posterior risk decreases for Nakagami distribution with a larger λ. These patterns are similar under each loss function. In comparison of priors it can be assessed that the performance of inverse gamma prior is better than that of uniform prior.

4.5 Loss Functions

The Bayes estimates are consistent under each loss function. However, the estimates under WBLF are associated with least amounts of posterior risks under each prior.

4.6 Final Remark

On the basis of above analysis it can be concluded that in order to estimate the parameter of Nakagami distribution under Bayesian framework, the use of inverse gamma prior under WBLF loss function can be preferred. The results are useful for the analysts looking to analyze the lifetime data and attenuation of wireless signals traversing multiple paths, deriving unit hydrographs in hydrology, medical imaging studies etc using Nakagami distribution under a Bayesian framework.

REFERENCES

[1]. Aalo, V. A., "Performance of maximal-ratio diversity systems in a correlated Nakagami fading environment", *IEEE Transactions on Communications*, 43(8), 2360-2369(1995).

[2] Lin, W. D. and Yang, H. S., (). Modeling the spatial-chromatic characteristics of images by Nakagami distribution.

[3] Radovan, R. and Jan, "Estimator comparison of the Nakagami parameter and its application in Echocardiography", Radio engineering, 13(1),8-13(2004).

[4] Sarkar, S., Goel, N. and Mathur, B., "Adequacy of Nakagami-m distribution function to derive GIUH", *Journal of Hydrologic Engineering*, 14(10), 1070- 1079(2009).

[5] Tsui, P., Huang, C. and Wang, S., "Use of Nakagami distribution and logistic compression in ultrasonic tissue characterization", *Journal of Medical and Biological Engineering*, 26(2), 69-73(2006).

[6] Zhang, Q. T., "A note on the estimation of Nakagami-m fading parameter", IEEE Communications Letters, 6(6), 237- 238(2002).

[7] Cheng, J. and Beaulieu, N. C., "Maximum-Likelihood based estimation of the Nakagami parameter IEEE Communications", 5(3),101-103(2001).

[8] Abid, A. and Kaveh, M., "Performance comparison of three different estimators for the Nakagami m parameter using Monte Carlo simulation", (2000).

[9] Smolikova, R., Wachowiak, P. M. and Zurada, M. J., "An information-theoretic approach to estimating ultrasound backscatter characteristics", Computers in *Biology and Medicine*, 34, 355-270(2004).

[10] Tsui, H., Chang, C., Ho, M. C., Lee, Y., Chen, Y., Chang, C., Huang, N. and Chang, K., "Use of Nakagami statistics and empirical mode decomposition for ultrasound tissue characterization by a nonfocused transducer", *Ultrasound in Medicine & Biology*, 35(12), 2055- 2068(2009).

[11] Karagiannidis, K. G., Zogas, A. D. and Kotsoupolas, A. S., "An efficient approach to multivariate Nakagami-m distribution using Green's matrix approximation", *IEEE Transactions on Wireless Communications*, 3(5), 883-889(2003).

[12] Schwartz, J. Gowdin, T. R. and Giles, E. D., "Improved maximum likelihood estimation of the shape parameter in the Nakagami distribution" *Mathematics Subjects Classification*, 62, 30-35(2011).

[13] Haq, A. and Dey, S., "Bayesian estimation of the Erlang distribution under different prior distributions", *Journal of Reliability and Statistical Studies*, 4(I), 1-30(2011).

[14] Dey, S., "Bayesian estimation of the parameter and reliability function of an inverse Rayleigh distribution", *Malaysian Journal of Mathematical Sciences*, 6(1), 113-124(2012).

[15] Feroze, N., "Estimation of scale parameter of inverse Gaussian distribution under a Bayesian framework using different loss functions", *Scientific Journal of Review*, 1(3), 39-52(2012).

[16] Feroze, N. and Aslam, M., "A note on Bayesian analysis of error function distribution under different loss functions", *International Journal of Probability and Statistics*, 1(5), 153-159(2012).

[17] Yahgmaei, F. Babanezhad, M. and Moghadam, S. O., "Bayesian estimation of the scale parameter of inverse Weibull distribution under the asymmetric loss functions", *Journal of Probability and Statistics*, (2013).